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1977 J. Phys. A: Math. Gen. 10 1665

(http://iopscience.iop.org/0305-4470/10/10/003)

J. Phys. A: Math. Gen., Vol. 10, No. 10, 1977. Printed in Great Britain. © 1977

An extended characterisation theorem for quantum logics

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Received 19 April 1977, in final form 6 July 1977

Abstract. Two theorems are proved. In the first properties of an important mapping from an orthocomplemented lattice to itself are studied. In the second we extend the characterisation theorem of Zierler to get a very useful theorem characterising orthomodular lattices. Since quantum logics are merely σ-complete orthomodular lattices, our principal result is, for application in quantum physics, a characterisation theorem for quantum logics.

1. Introduction

Formal quantum theories defined on both Hilbert spaces and C^*-algebras have their foundations built on quantum logics (for a full discussion see Mukherjee 1977). Varadarajan (1968) and numerous other workers have studied quantum logics in considerable detail. In this paper we prove a characterisation theorem which sums up very neatly almost all the important properties of quantum logics. On our way to proving this theorem, we study the properties of an important mapping from an orthocomplemented lattice to itself. A number of related results are also proved in the form of lemmas and corollaries.

2. Formalities

We use the following definitions and notation:

Definition 1. A *lattice* is a partially ordered set (a partially ordered set is hereafter called a *poset* for brevity), in which any two elements, say a and b, have both a least upper bound called the *join* of the two elements denoted by $a \vee b$ and a greatest lower bound called the *meet* of the two elements and denoted by $a \wedge b$. If S is a subset of a lattice, then the join and the meet of the subset if they exist are denoted by $\vee S$ and $\wedge S$ respectively. A lattice is said to be *complete* if every subset S of the lattice has both a join and a meet. A lattice is said to be *σ-complete* if every countable subset of the lattice has both a join and a meet.

Definition 2. A lattice L is said to have a *universal upper bound* (*universal lower bound*) if $\vee L (\wedge L)$ exists in L, in which case it is denoted by $I(O)$:

Definition 3. A lattice L with universal bounds I and O is called a *complemented lattice* if for each $a \in L$, there exists an $a' \in L$ such that

$$a \vee a' = I \qquad \text{and} \qquad a \wedge a' = O.$$

Definition 4. A complemented lattice is called *orthocomplemented* if the complementation $a \mapsto a'$ satisfies the following further requirements:

$$(a')' = a \qquad \forall a \in L$$

and

$$a \leqslant b \Rightarrow b' \leqslant a'.$$

Definition 5. An order-reversing bijection from a lattice to itself is called a *dual automorphism* of the lattice.

Throughout this work we use the symbol $\Rightarrow$ to denote 'implies'. In particular $P \Rightarrow Q$ means that whenever P is true, so also is Q.

We expect the reader to be familiar with the following lemma (a proof can be found in McLane and Birkhoff 1967).

Lemma 2.1. Every lattice L has the following properties:
Idempotent:

$$a \vee a = a;\ a \wedge a = a \qquad \forall a \in L$$

Commutative:

$$a \vee b = b \vee a;\ a \wedge b = b \wedge a \qquad \forall a, b \in L$$

Associative:

$$(a \vee b) \vee c = a \vee (b \vee c);\ (a \wedge b) \wedge c = a \wedge (b \wedge c) \qquad \forall a, b, c \in L$$

Absorption:

$$a = a \vee (a \wedge b) = a \wedge (a \vee b) \qquad \forall a, b \in L$$

Consistency:

$$a \leqslant b \Leftrightarrow a \wedge b = a \text{ and } a \vee b = b \qquad \forall a, b \in L$$

Isotone:

$$b \leqslant c \Rightarrow a \vee b \leqslant a \vee c \text{ and } a \wedge b \leqslant a \wedge c \qquad \forall a, b, c \in L$$

Distributive inequalities:

$$\left.\begin{array}{l} a \wedge (b \vee c) \geqslant (a \wedge b) \vee (a \wedge c) \\ a \vee (b \wedge c) \leqslant (a \vee b) \wedge (a \vee c) \end{array}\right. \qquad \forall a, b, c \in L$$

Modular inequality:

$$a \leqslant c \Rightarrow a \vee (b \wedge c) \leqslant (a \vee b) \wedge c \qquad \forall a, b, c \in L.$$

We can now define:

Definition 6. A lattice L is said to be *distributive* if the distributive inequalities are strict equalities in it.

Definition 7. A lattice L is said to be *modular* if the modular inequality is a strict equality in it.

Definition 8. A map $\psi : L \to L$ is called a *join morphism* on L if

$$\psi(a \vee b) = \psi(a) \vee \psi(b) \qquad \forall a, b \in L.$$

If in addition L has a universal lower bound and

$$\wedge \psi(L) = O,$$

ψ is called a *hemimorphism* on L.

Lemma 2.2. In an orthocomplemented lattice L the complementation $'$ is a dual automorphism of L.

Proof. Let $a' = b'$. Then $(a')' = a = (b')' = b$. Thus $'$ is injective.
Since $a = (a')' \forall a \in L$, $'$ is surjective.
Let $a \vee b = c$. Then $a \leqslant c$ and $b \leqslant c \Rightarrow a' \geqslant c'$ and $b' \geqslant c' \Rightarrow c' \leqslant a' \wedge b'$, that is, $(a \vee b)' \leqslant a' \wedge b'$.
Let $a' \wedge b' = d$. Then $d \leqslant a'$ and $d \leqslant b' \Rightarrow a \leqslant d'$ and $b \leqslant d' \Rightarrow (a \vee b) \leqslant d' \Rightarrow d = a' \wedge b' \leqslant (a \vee b)'$.
Hence $(a \vee b)' = a' \wedge b'$.
Similarly $(a \wedge b)' = a' \vee b'$.
This completes the proof.

Corollary 2.2.1. Let L be an orthocomplemented lattice. Let S be a subset of L. Then
(i) $\vee S$ exists implies $\wedge S'$ exists (where $S' = \{x' : x \in S\}$ and

$$(\vee S)' = \wedge S'.$$

(ii) $\wedge S$ exists implies $\vee S'$ exists and

$$(\wedge S)' = \vee S'.$$

Proof. Suppose $\vee S$ exists and $\vee S = c$. This implies $x \leqslant c \ \forall x \in S \Rightarrow c' \leqslant x' \ \forall x \in S \Rightarrow c' \leqslant \wedge S'$.
Let d be a lower bound for S'. Then $d \leqslant x' \ \forall x' \in S' \Rightarrow x \leqslant d' \ \forall x \in S \Rightarrow c = \vee S \leqslant d' \Rightarrow d \leqslant c'$.
Thus c' is a lower bound for S' which is greater than or equal to every lower bound for S'. Hence c' is the meet of S'. Thus $\wedge S'$ exists and

$$\wedge S' = (\vee S)'.$$

The proof of (ii) is similar. We are finished.

Corollary 2.2.2. Let $\psi : L_1 \to L_2$ be a dual isomorphism. Let S be a subset of L_1. Then
(i) the existence of either $\vee S$ or $\wedge \psi(S)$ implies the existence of both and $\vee S = \wedge \psi(S)$;
(ii) the existence of either $\wedge S$ or $\vee \psi(S)$ implies the existence of both and $\wedge S = \vee \psi(S)$.

Proof. The proof is exactly similar to that of the preceding corollary.

3. The two theorems

Throughout this section L denotes an orthocomplemented lattice. Let $c \in L$. The mapping $L \to L$ defined by $x \mapsto (x \vee c') \wedge c$ plays a very important part in the theory of quantum logics and in particular their relationship to Baer $*$-semigroups (Foulis 1960). In the first theorem we collect together the important properties of this mapping.

Theorem 3.1. Let L be an orthocomplemented lattice. Let $c \in L$. Then $\phi_c : L \to L$ defined by $\phi_c(x) = (x \vee c') \wedge c$ has the following properties:

(i) $\phi_c(O) = O$,

(ii) $\phi_c \circ \phi_c = \phi_c$,

(iii) $x \in L_c = \{x : x \in L \,\&\, O \leqslant x \leqslant c\} \Rightarrow x \leqslant \phi_c(x)$,

$$x' \wedge (\phi_c(x)) = O \text{ and } x' \wedge c = (\phi_c(x))' \wedge c,$$

(iv) a sufficient condition that ϕ_c is a hemimorphism on L is that

$$x \notin L_c \Rightarrow x' \wedge c \in \phi_c(L) \text{ and}$$

(v) a necessary condition that ϕ_c is a hemimorphism on L is that

$$c = \phi_c(x) \vee ((\phi_c(x))' \wedge c) = \phi_c(x) \vee \phi_c(x').$$

Proof. (i) $\phi_c(O) = (O \vee c') \wedge c = c' \wedge c = O$.

(ii) Let $x \in L$ and let $h = \phi_c(x) = (x \vee c') \wedge c$.
Then $h \leqslant x \vee c' \Rightarrow h \vee c' \leqslant x \vee c' \Rightarrow (h \vee c') \wedge c \leqslant (x \vee c') \wedge c = h$.
On the other hand

$$h \leqslant h \vee c' \text{ and } h \leqslant c \Rightarrow h \leqslant (h \vee c') \wedge c.$$

Hence $\phi_c \circ \phi_c(x) = \phi_c(x)$ $\forall x \in L$, or in other words

$$\phi_c \circ \phi_c = \phi_c.$$

(iii) $x \in L_c \Rightarrow x \leqslant c \Rightarrow x \leqslant (x \vee c') \wedge c = \phi_c(x) = h$.
$x' \wedge \phi_c(x) = x' \wedge (x \vee c') \wedge c = (x' \wedge c) \wedge (x \vee c') = (x' \wedge c) \wedge (x' \wedge c)' = O$.
$h \leqslant x \vee c' \Rightarrow h \vee c' \leqslant x \vee c'$, but $x \leqslant h \Rightarrow x \vee c' \leqslant h \vee c'$.
Hence $x \vee c' = h \vee c' \Rightarrow x' \wedge c = h' \wedge c = (\phi_c(x))' \wedge c$.

(iv) Let $x \notin L_c \Rightarrow x' \wedge c \in \phi_c(L)$.
Then if $x \notin L_c$, there exists a $z \in L$ such that

$$\phi_c(z) = x' \wedge c \Rightarrow ((x' \wedge c) \vee c') \wedge c = \phi_c \circ \phi_c(z) = \phi_c(z) = x' \wedge c.$$

In view of (iii) above we have $\forall x \in L$,

$$x' \wedge c = ((x' \wedge c) \vee c') \wedge c.$$

Now let $g = \phi_c(x) \vee \phi_c(y) = ((x \vee c') \wedge c) \vee ((y \vee c') \wedge c) \leqslant (x \vee y \vee c') \wedge c = \phi_c(x \vee y)$.
Hence $(x \vee c') \wedge c \leqslant g \Rightarrow g' \leqslant (x' \wedge c) \vee c' \Rightarrow (g' \vee c') \wedge c \leqslant ((x' \wedge c) \vee c') \wedge c = x' \wedge c$.
Similarly $(g' \vee c') \wedge c \leqslant y' \wedge c$.
Hence $(g' \vee c') \wedge c \leqslant x' \wedge y' \wedge c \Rightarrow (g \wedge c) \vee c' \geqslant x \vee y \vee c' \Rightarrow ((g \wedge c) \vee c') \wedge c = g \wedge c \geqslant (x \vee y \vee c') \wedge c \Rightarrow g = g \wedge c \geqslant (x \vee y \vee c') \wedge c$.
Hence $\phi_c(x \vee y) = \phi_c(x) \vee \phi_c(y)$.

(v) Let ϕ_c be a hemimorphism. Then

$$\phi_c(x \vee x') = (x \vee x' \vee c') \wedge c = (I \vee c') \wedge c = c = \phi_c(x) \vee \phi_c(x').$$

Let $\phi_c(x)=h$, then $\phi_c(h \vee h')=c=\phi_c(h) \vee \phi_c(h')=\phi_c \circ \phi_c(x) \vee ((h' \vee c') \wedge c)$.
But $\phi_c \circ \phi_c(x)=\phi_c(x)=h$, and $h \leqslant c \Rightarrow c' \leqslant h' \Rightarrow h' \vee c'=h'$.
Hence $c=h \vee (h' \wedge c)=\phi_c(x) \vee \phi_c(x')$.
This completes the proof of our theorem.

Remark 3.1.1. We shall soon see that in a quantum logic

$$x \in L_c \Rightarrow x=\phi_c(x)=(x \vee c') \wedge c$$

and

$$c=(\phi_{x'}(c'))'=((c' \vee x'') \wedge x')'=x \vee (c \wedge x').$$

Now the preceding theorem tells us that in any orthocomplemented lattice

$$x \in L_c \Rightarrow x \leqslant \phi_c(x),\ c=\phi_c(x) \vee (x' \wedge c)$$

and $x' \wedge \phi_c(x)=O$, provided

$$x \notin L_c \Rightarrow x' \wedge c \in \phi_c(L).$$

If the orthocomplemented lattice has the property that

$$a \leqslant b \quad \text{and} \quad a' \wedge b=O \Rightarrow a=b,$$

then we will have $\phi_c(x)=x$ and $c=x \vee (x' \wedge c)$, in other words, our orthocomplemented lattice, if σ-complete, will become a quantum logic. Thus we have rediscovered a definitive property of quantum logics given by Zierler (1961). In doing so we had to assume that the orthocomplemented lattice satisfied the sufficient condition that ϕ_c is a hemimorphism. It is easy to see that this condition implies that $L_c=\phi_c(L)$ and if this new condition is satisfied for every $c \in L$, our orthocomplemented lattice, if σ-complete, is a quantum logic. We thus have a new definitive property of a quantum logic. We collect together all the definitive properties of an orthomodular lattice in our next theorem.

Theorem 3.2. An orthocomplemented lattice L is said to be *orthomodular* or *weakly modular* if it has anyone and, therefore, everyone of the following properties:

(i) $a, b \in L$ and $a \leqslant b \Rightarrow a \vee (a' \wedge b)=b$.
(ii) $a, b \in L$ and $a \leqslant b \Rightarrow \exists c \in L$ such that $c \leqslant a'$ and $a \vee c=b$.
(iii) $a, b \in L$, $a \leqslant b$ and $a' \wedge b=O \Rightarrow b=a$.
(iv) $L_c=\phi_c(L)\ \forall c \in L$.
(v) $a, b, c \in L$, $a \leqslant c$ and $b \leqslant c' \Rightarrow (a \vee b) \wedge c=a$.
(vi) $a, b, c \in L$, $a \leqslant c$ and $a \leqslant b' \Rightarrow (a \vee b) \wedge c=a \vee (b \wedge c)$.

Proof. We shall first prove the equivalence of the first five by proving (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(i). Then we shall prove that (iii) and (v)$\Rightarrow$(vi)$\Rightarrow$(iv), thus completing the proof.
Proof of (i)$\Rightarrow$(ii): Take $c=a' \wedge b$, clearly $c \leqslant a'$ and $a \vee c=b$.
Proof of (ii)$\Rightarrow$(iii): From (ii) there exists a $c \leqslant a'$ such that $a \vee c=b$.
Clearly $c \leqslant b \Rightarrow c \wedge b=c$, but $c \wedge b \leqslant a' \wedge b=O \Rightarrow c=O \Rightarrow b=a \vee c=a \vee O=a$.
Proof of (iii)$\Rightarrow$(iv): From Theorem 3.1 $a \in L_c \Rightarrow a \leqslant \phi_c(a)$ and $a' \wedge \phi_c(a)=O$.
Hence from (iii) $a=\phi_c(a) \Rightarrow L_c \subset \phi_c(L)$.
But $x \in L \Rightarrow \phi_c(x)=(x \vee c') \wedge c \leqslant c \Rightarrow \phi_c(L) \subset L_c$.
Thus $L_c=\phi_c(L)\ \forall c \in L$.

Proof of (iv)$\Rightarrow$(v): $L_c = \phi_c(L) \Rightarrow$ if $a \leqslant c$, then there is an element $d \in L$ such that $a = \phi_c(d) \Rightarrow \phi_c(a) = \phi_c \circ \phi_c(d) = \phi_c(d) = a$, that is

$$a = (a \vee c') \wedge c.$$

If in addition $b \leqslant c'$, then $(a \vee b) \wedge c \leqslant (a \vee c') \wedge c = a$. On the other hand $a \leqslant a \vee b$ and $a \leqslant c \Rightarrow a \leqslant (a \vee b) \wedge c$. Thus $(a \vee b) \wedge c = a$.
Proof of (v)$\Rightarrow$(i): $a \leqslant b \Rightarrow b' \leqslant a'$ and $a \leqslant a'' = a \Rightarrow (b' \vee a) \wedge a' = b' \Rightarrow b = a \vee (a' \wedge b)$.
Proof of (iii) and (v)$\Rightarrow$(vi): Let $a \leqslant c$ and $a \leqslant b'$. From modular inequality

$$a \vee (b \wedge c) \leqslant (a \vee b) \wedge c.$$

But

$$\begin{aligned}(a \vee (b \wedge c))' \wedge (a \vee b) \wedge c \\ &= a' \wedge (b' \vee c') \wedge (a \vee b) \wedge c = (b' \vee c') \wedge ((a \vee b) \wedge a') \wedge c \\ &= (b' \vee c') \wedge (b \wedge c) = O,\end{aligned}$$

where we have used (v) in deducing that $(a \vee b) \wedge a' = b$.
Hence $a \vee (b \wedge c) = (a \vee b) \wedge c$.
Proof of (vi)$\Rightarrow$(iv): By taking $b = c'$ in (vi) we see that $a \leqslant c$ and $a \leqslant b' = c$ are the same condition and both are satisfied, hence

$$(a \vee c') \wedge c = a \vee (c' \wedge c) = a \vee O = a.$$

This finally completes the proof of our theorem.

Remark 3.2.0. Theorem 3.2 being a characterisation theorem looks partly like a definition and indeed it gives six equivalent definitions of an orthomodular lattice and proves the equivalence of the different definitions.

Definition 9. A σ-complete orthomodular lattice is called a *quantum logic*.

Corollary 3.2.1. c which exists by virtue of theorem 3.2(ii) is unique.

Proof. $a \vee c = b$ and $c \leqslant a' \Rightarrow c \leqslant a' \wedge b$, but $c' \wedge a' \wedge b = (a \vee c)' \wedge b = b' \wedge b = O \Rightarrow c = a' \wedge b$. Thus c is uniquely determined by the last equality and we are finished.

Corollary 3.2.2. Let L be a weakly modular orthocomplemented lattice. Then $L[a, b] = \{x : x \in L \,\&\, a \leqslant x \leqslant b\}$ is a weakly modular orthocomplemented lattice with universal bounds a and b in which the orthocomplementation $\times : x \mapsto x^\times$ is defined by $x^\times = a \vee (x' \wedge b) = (a \vee x') \wedge b$ where $'$ is the orthocomplementation in L.

Proof. That a, b are universal bounds in $L[a, b]$ is clear from the definition of $L[a, b]$. Now $x \in L[a, b] \Rightarrow a \vee (x' \wedge b) = (a \vee x') \wedge b$ because $a \leqslant b$ and $a \leqslant (x')' = x$ and theorem 3.2(vi) applies.

We now prove that $x, y \in L[a, b]$, $x \leqslant y \Rightarrow y = x \vee (x^\times \wedge y)$.
$x^\times \wedge y = (a \vee x') \wedge b \wedge y = (a \vee x') \wedge y = a \vee (x' \wedge y)$ by theorem 3.2(vi).
Hence $x \vee (x^\times \wedge y) = x \vee a \vee (x' \wedge y) = x \vee (x' \wedge y) = y$ by theorem 3.2(i).
Thus the corollary follows from theorem 3.2(i).

Remark 3.2.3. Our theorem with corollaries proves with substantially greater ease the properties established in Varadarajan (1968, § 1 of chap. 6) and much more.

Remark 3.2.4. A somewhat weaker sufficient condition that ϕ_c of theorem 3.1 is a hemimorphism is as follows:

$$x \in L_c \cup L_{c'} \Rightarrow x \wedge c' \in \phi_c(L) \qquad \text{and} \qquad y, z \in \phi_c(L) \Rightarrow y \vee z \in \phi_c(L).$$

The proof is very similar to that given in theorem 3.1. We note that an orthocomplemented lattice satisfying the sufficient condition of theorem 3.1 is weakly modular whereas one satisfying the condition given above may not be so.

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